

Tetrahedra on deformed spheres and integral group cohomology

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Abstract

We show that for every injective continuous map $f : S^2 \rightarrow \mathbb{R}^3$ there are four distinct points in the image of f such that the convex hull is a tetrahedron with the property that two opposite edges have the same length and the other four edges are also of equal length. This result represents a partial result for the topological Borsuk problem for \mathbb{R}^3 . Our proof of the geometrical claim, via Fadell–Husseini index theory, provides an instance where arguments based on group cohomology with integer coefficients yield results that cannot be accessed using only field coefficients.

1 Introduction

The motivation for the study of the existence of particular types of tetrahedra on deformed 2-spheres is twofold. The topological Borsuk problem, as considered in [7], along with the square peg problem [6] inspire the search for possible polytopes with nice metric properties whose vertices lie on the continuous images of spheres. Beyond their intrinsic interest, these problems can be used as testing grounds for tools from equivariant topology, e.g. for comparing the strength of Fadell–Husseini index theory with ring resp. field coefficients.

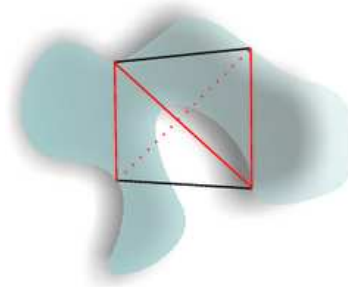


Figure 1: D_8 -invariant tetrahedra on deformed sphere S^2

The following theorem will be proved through the use of Fadell–Husseini index theory with coefficients in the ring \mathbb{Z} . It is also going to be demonstrated that Fadell–Husseini index theory with coefficients in field \mathbb{F}_2 has no power in this instance (Section 4.1).

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Theorem 1.1. *Let $f : S^2 \rightarrow \mathbb{R}^3$ an injective continuous map. Then its image contains vertices of a tetrahedron that has the symmetry group D_8 of a square. That is, there are four distinct points ξ_1, ξ_2, ξ_3 and ξ_4 on S^2 such that*

$$d(f(\xi_1), f(\xi_2)) = d(f(\xi_2), f(\xi_3)) = d(f(\xi_3), f(\xi_4)) = d(f(\xi_4), f(\xi_1))$$

and

$$d(f(\xi_1), f(\xi_3)) = d(f(\xi_2), f(\xi_4)).$$

Remark 1.2. The proof is not going to use any properties of \mathbb{R}^3 except that it is a metric space. Thus in the statement of the theorem, \mathbb{R}^3 can be replaced by any metric space (M, d) .

Remark 1.3. Unfortunately, the methods used for the proof of Theorem 1.1 do not imply any conclusion when applied to the square peg problem (see Section 4.2). On the other hand, if the square peg problem could be solved for the continuous Jordan curves, then it would imply the result of Theorem 1.1.

2 Introducing the equivariant question

Let $f : S^2 \rightarrow \mathbb{R}^3$ be an injective continuous map. Denote by D_8 the symmetry group of a square, that is, the 8-element dihedral group $D_8 = \langle \omega, j \mid \omega^4 = j^2 = 1, \omega j = j\omega^3 \rangle$.

A few D_8 -representations.

The vector spaces

$$\begin{aligned} U_4 &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}, \\ U_2 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 0\} \end{aligned}$$

are D_8 -representations with actions given by

(a) for $(x_1, x_2, x_3, x_4) \in U_4$:

$$\omega \cdot (x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, x_1), \quad j \cdot (x_1, x_2, x_3, x_4) = (x_3, x_2, x_1, x_4),$$

(b) for $(x_1, x_2) \in U_2$:

$$\omega \cdot (x_1, x_2) = (x_2, x_1), \quad j \cdot (x_1, x_2) = (x_2, x_1),$$

The configuration space.

Let $X = S^2 \times S^2 \times S^2 \times S^2$ and let Y be the subspace given by

$$Y = \{(x, y, x, y) \mid x, y \in S^2\} \approx S^2 \times S^2.$$

The configuration space to be considered is the space

$$\Omega := X \setminus Y.$$

Let a D_8 -action on X be induced by

$$\omega \cdot (\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_2, \xi_3, \xi_4, \xi_1), \quad j \cdot (\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_4, \xi_3, \xi_2, \xi_1),$$

for $(\xi_1, \xi_2, \xi_3, \xi_4) \in X$.

A test map.

Let $\tau : \Omega \rightarrow U_4 \times U_2$ be a map defined for $(\xi_1, \xi_2, \xi_3, \xi_4) \in X$ by

$$\tau(\xi_1, \xi_2, \xi_3, \xi_4) = (d_{12} - \frac{\Delta}{4}, d_{23} - \frac{\Delta}{4}, d_{34} - \frac{\Delta}{4}, d_{41} - \frac{\Delta}{4}) \times (d_{13} - \frac{\Phi}{2}, d_{24} - \frac{\Phi}{2}) \quad (1)$$

where $d_{ij} := d(f(\xi_i), f(\xi_j))$ and

$$\Delta = d_{12} + d_{23} + d_{34} + d_{14}, \quad \Phi = d_{13} + d_{24}.$$

With the D_8 -actions introduced above the test map τ is D_8 -equivariant. The following proposition connects our set-up with the tetrahedron problem.

Proposition 2.1. *If there is no D_8 -equivariant map*

$$\alpha : \Omega \rightarrow (U_4 \times U_2) \setminus (\{\mathbf{0}\} \times \{\mathbf{0}\}) \quad (2)$$

then Theorem 1.1 follows.

Proof. If there is no D_8 -equivariant map $\Omega \rightarrow (U_4 \times U_2) \setminus (\{\mathbf{0}\} \times \{\mathbf{0}\})$, then for every continuous embedding $f : S^2 \rightarrow \mathbb{R}^3$ there is a point $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega = X \setminus Y$ such that

$$\tau(\xi_1, \xi_2, \xi_3, \xi_4) = (\mathbf{0}, \mathbf{0}) \in U_4 \times U_2. \quad (3)$$

From (3) we conclude that

$$d_{12} = d_{23} = d_{34} = d_{14} = \frac{\Delta}{4} \quad \text{and} \quad d_{13} = d_{24} = \frac{\Phi}{2}. \quad (4)$$

It only remains to prove that all four points are different. Since $(\xi_1, \xi_2, \xi_3, \xi_4) \notin Y$ we have $\xi_1 \neq \xi_3$ or $\xi_2 \neq \xi_4$. By symmetry we may assume that $\xi_1 \neq \xi_3$. The map f is injective, therefore $f(\xi_1) \neq f(\xi_3)$ and consequently $d_{13} \neq 0$. Now

$$d_{13} \neq 0 \Rightarrow d_{24} \neq 0 \Rightarrow f(\xi_1) \neq f(\xi_3), f(\xi_2) \neq f(\xi_4) \Rightarrow \xi_1 \neq \xi_3, \xi_2 \neq \xi_4.$$

Let us assume, without loss of generality, that $\xi_1 = \xi_2$. Then $d_{12} = d_{23} = d_{34} = d_{14} = 0$, which implies that $d_{13} \leq d_{12} + d_{23} = 0$. This yield a contradiction to $d_{13} \neq 0$. Thus $\xi_1 \neq \xi_2$. \square

By Proposition 2.1, Theorem 1.1 is a consequence of the following topological result.

Theorem 2.2. *There is no D_8 -equivariant map $\Omega \rightarrow S(U_4 \times U_2)$.*

3 Proof of Theorem 2.2

The proof is going to be conducted through a comparison of the Serre spectral sequences with \mathbb{Z} -coefficients of the Borel constructions associated with the spaces Ω and $S(U_4 \times U_2)$ and the subgroup $\mathbb{Z}_4 = \langle \omega \rangle$ of D_8 . In other words, we determine the \mathbb{Z}_4 Fadell–Husseini index of these spaces living in $H_4^*(\mathbb{Z}_4; \mathbb{Z}) = \mathbb{Z}[U]/4U$, $\deg U = 2$.

The Fadell–Husseini index of a G -space X is the kernel of the map $\pi_X^* : H^*(BG, \mathbb{Z}) \rightarrow H^*(X \times_G EG, \mathbb{Z})$ induced by the projection $\pi_X : X \times_G EG \rightarrow BG$. If $E_*^{*,*}$ denotes the Serre spectral sequence of the Borel construction of X , then the homomorphism π_X^* can be presented as the composition

$$H^*(BG, \mathbb{Z}) \rightarrow E_2^{*,0} \rightarrow E_3^{*,0} \rightarrow E_4^{*,0} \rightarrow \dots \rightarrow E_\infty^{*,0} \subseteq H^*(X \times_G EG, \mathbb{Z}). \quad (5)$$

Since the E_2 -term of the spectral sequence is given by $E_2^{p,q} = H^p(BG, H^q(X, \mathbb{Z}))$ the first step in the computation of the index is study of the cohomology $H^*(X, \mathbb{Z})$ as a G -module (Section 3.2). The final step is explicit description of non-zero differentials in the spectral sequence and application of the presentation (5) of the homomorphism π_X^* (Section 3.3).

3.1 The Index of $S(U_4 \times U_2)$

Let V^1 be the 1-dimensional complex representation of \mathbb{Z}_4 induced by $1 \mapsto e^{i\pi/2}$. Then the representation $U_4 \subset \mathbb{R}^4$ seen as a \mathbb{Z}_4 -representation decomposes into a sum of two irreducible \mathbb{Z}_4 -representations

$$U_4 = \text{span}\{(1, 0, -1, 0), (0, 1, 0, -1)\} \oplus \text{span}\{(1, -1, 1, -1)\} \cong V^1 \oplus U_2.$$

Here “span” stands for all \mathbb{R} -linear combinations of the given vectors. It can be also seen that

$$U_4 \times U_2 \cong V^1 \oplus U_2 \oplus U_2 \cong V^1 \oplus (V^1 \otimes V^1).$$

Following [1, Section 8, p. 271 and Appendix, page 285] we deduce the total Chern class of the \mathbb{Z}_4 -representation $U_4 \times U_2$

$$c(U_4 \times U_2) = c(V^1) \cdot c(V^1 \otimes V^1)$$

and consequently the top Chern class

$$c_2(U_4 \times U_2) = c_1(V^1) \cdot c_1(V^1 \otimes V^1) = c_1(V^1) \cdot (c_1(V^1) + c_1(V^1)) = 2U^2 \in H^*(\mathbb{Z}_4; \mathbb{Z}).$$

The \mathbb{Z}_4 -index of the sphere $S(U_4 \times U_2)$ is given by [2, Proposition 3.11] as

$$\text{Index}_{\mathbb{Z}_4, \mathbb{Z}} S(U_4 \times U_2) = \langle 2U^2 \rangle. \quad (6)$$

3.2 The cohomology $H^*(\Omega; \mathbb{Z})$ as a \mathbb{Z}_4 -module

The cohomology is going to be determined via Poincaré–Lefschetz duality and an explicit study of cell structures for the spaces X and Y .

Poincaré–Lefschetz duality [5, Theorem 70.2, page 415] implies that

$$H^*(\Omega; \mathbb{Z}) = H^*(X \setminus Y; \mathbb{Z}) \cong H_{8-*}(X, Y; \mathbb{Z}) \quad (7)$$

and therefore we analyze the homology of the pair (X, Y) .

The long exact sequence in homology of the pair (X, Y) yields that the possibly non-zero homology groups of the pair (X, Y) with \mathbb{Z} -coefficients are

$$H_i(X, Y; \mathbb{Z}) = \begin{cases} \mathbb{Z}[\mathbb{Z}_4]/\text{im}\Phi, & i = 2 \\ \ker \Phi, & i = 3 \\ \mathbb{Z}[\mathbb{Z}_4] \oplus \mathbb{Z}[\mathbb{Z}_4/\mathbb{Z}_2]/\text{im}\Psi, & i = 4 \\ \ker \Psi, & i = 5 \\ \mathbb{Z}[\mathbb{Z}_4], & i = 6 \\ \mathbb{Z}, & i = 8 \end{cases}$$

Thus explicit formulas for the maps $\Phi : H_2(Y; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z})$ and $\Psi : H_4(Y; \mathbb{Z}) \rightarrow H_4(X; \mathbb{Z})$, induced by the inclusion $Y \subset X$, are needed in order to determine the homology $H_*(X, Y; \mathbb{Z})$ and its exact \mathbb{Z}_4 -module structure.

Let $x_1, x_2, x_3, x_4 \in H_2(X; \mathbb{Z})$ be generators carried by individual copies of S^2 in the product $X = S^2 \times S^2 \times S^2 \times S^2$. The generator of the group $\mathbb{Z}_4 = \langle \omega \rangle$ acts on this basis of $H_2(X; \mathbb{Z})$ by $\omega \cdot x_i = x_{i+1}$ where $x_5 = x_1$. Then by $x_i x_j \in H_4(X; \mathbb{Z})$, $i \neq j$, we denote the generator carried by the product of i -th and j -th copy of S^2 in X . Since ω is not changing the orientation the action on $H_4(X; \mathbb{Z})$ is described by

$$x_1 x_2 \xrightarrow{\cdot \omega} x_2 x_3 \xrightarrow{\cdot \omega} x_3 x_4 \xrightarrow{\cdot \omega} x_1 x_4 \quad \text{and} \quad x_1 x_3 \xrightarrow{\cdot \omega} x_2 x_4.$$

Let similarly $y_1, y_2 \in H_2(X; \mathbb{Z})$ be generators carried by individual copies of S^2 in the product $Y = S^2 \times S^2$. Then $\omega \cdot y_1 = y_2$ and $\omega \cdot y_2 = y_1$. Again $y_1 y_2$ denotes the generator of $H_4(Y; \mathbb{Z})$ and $\omega \cdot y_1 y_2 = y_1 y_2$. Note that ω preserves the orientations of X and Y and therefore acts trivially on $H_8(X; \mathbb{Z})$ and on $H_4(Y; \mathbb{Z})$.

The inclusion $Y \subset X$ induces a map in homology $H_*(X; \mathbb{Z}) \subset H_*(Y; \mathbb{Z})$, which in dimensions 2 and 4 is given by

$$y_1 \mapsto x_1 + x_3, \quad y_2 \mapsto x_2 + x_4,$$

$$y_1 y_2 \mapsto x_1 x_2 + x_2 x_3 + x_3 x_4 + x_1 x_4.$$

This can be seen from the dual cohomology picture: An element is mapped to a sum of generators intersecting its image, with appropriately attached intersection numbers. Thus Φ and Ψ are injective and

$$\text{im}\Phi = \langle x_1 + x_3, x_2 + x_4 \rangle, \quad \text{im}\Psi = \langle x_1x_2 + x_2x_3 + x_3x_4 + x_1x_4 \rangle.$$

Let $N = \mathbb{Z} \oplus \mathbb{Z}$ be the \mathbb{Z}_4 -representation given by $\omega \cdot (a, b) = (b, -a)$, while M denotes the representation $\mathbb{Z}[\mathbb{Z}_4]/(1+\omega+\omega^2+\omega^3)\mathbb{Z}$. Then the non-trivial cohomology of the space $X \setminus Y$, as a \mathbb{Z}_4 -module via the isomorphism (7), is given by

$$H^i(\Omega; \mathbb{Z}) = \begin{cases} N, & i = 6 \\ M \oplus \mathbb{Z}[\mathbb{Z}_4/\mathbb{Z}_2], & i = 4 \\ \mathbb{Z}[\mathbb{Z}_4], & i = 2 \\ \mathbb{Z}, & i = 0 \end{cases} \quad (8)$$

3.3 The Serre spectral sequence of the Borel construction $\Omega \times_{\mathbb{Z}_4} E\mathbb{Z}_4$

The Serre spectral sequence associated to the fibration $\Omega \rightarrow \Omega \times_{\mathbb{Z}_4} E\mathbb{Z}_4 \rightarrow B\mathbb{Z}_4$ is a spectral sequence with non-trivial local coefficients, since $\pi_1(B\mathbb{Z}_4) = \mathbb{Z}_4$ acts non-trivially (8) on the cohomology $H^*(\Omega; \mathbb{Z})$. The first step in the study of such a spectral sequence is to understand the $H^*(\mathbb{Z}_4; \mathbb{Z})$ -module structure on the rows of its E_2 -term.

The E_2 -term of the sequence is given by

$$E_2^{p,q} = \begin{cases} H^p(\mathbb{Z}_4, N), & q = 6 \\ H^p(\mathbb{Z}_4, M) \oplus H^p(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4/\mathbb{Z}_2]), & q = 4 \\ H^p(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4]), & q = 2 \\ H^p(\mathbb{Z}_4; \mathbb{Z}), & q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.1. $H^p(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4]) = \begin{cases} \mathbb{Z}, & p = 0 \\ 0, & p > 0 \end{cases}$

and multiplication by $U \in H^p(\mathbb{Z}_4; \mathbb{Z})$ is trivial, $U \cdot H^p(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4]) = 0$.

For the proof one can consult [3, Exercise 2, page 58].

Lemma 3.2. $H^*(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4/\mathbb{Z}_2]) \cong H^*(\mathbb{Z}_2; \mathbb{Z})$, where the module structure is given by the restriction homomorphism $\text{res}_{\mathbb{Z}_2}^{\mathbb{Z}_4} : H^*(\mathbb{Z}_4; \mathbb{Z}) \rightarrow H^*(\mathbb{Z}_2; \mathbb{Z})$.

In other words, if we denote $H^*(\mathbb{Z}_2; \mathbb{Z}) = \mathbb{Z}[T]/2T$, $\deg T = 2$, then $\text{res}_{\mathbb{Z}_2}^{\mathbb{Z}_4}(U) = T$ and consequently:

- (A) $H^*(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4/\mathbb{Z}_2])$ is generated by one element of degree 0 as a $H^*(\mathbb{Z}_4; \mathbb{Z})$ -module, and
- (B) multiplication by U in $H^*(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4/\mathbb{Z}_2])$ is an isomorphism, while multiplication by $2U$ is zero.

The proof is a direct application of Shapiro's lemma [3, (6.3), page 73] and a small part of the restriction diagram [2, Section 4.5.2].

Lemma 3.3. Let $\Lambda \in H^*(\mathbb{Z}_4, M)$ denote an element of degree 1 such that $4\Lambda = 0$. Then $H^*(\mathbb{Z}_4, M) \cong H^*(\mathbb{Z}_4; \mathbb{Z}) \cdot \Lambda$ as an $H^*(\mathbb{Z}_4; \mathbb{Z})$ -module.

Proof. The short exact sequence of \mathbb{Z}_4 -modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1+\omega+\omega^2+\omega^3} \mathbb{Z}[\mathbb{Z}_4] \longrightarrow M \longrightarrow 0$$

induces a long exact sequence in cohomology [3, Proposition 6.1, page 71], which is natural with respect to $H^*(\mathbb{Z}_4; \mathbb{Z})$ -module multiplication. Since $\mathbb{Z}[\mathbb{Z}_4]$ is a free module we get enough zeros to recover the

information we need:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(\mathbb{Z}_4; \mathbb{Z}) & \longrightarrow & H^0(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4]) & \longrightarrow & H^0(\mathbb{Z}_4, M) \longrightarrow H^1(\mathbb{Z}_4; \mathbb{Z}) \longrightarrow \\
& & \mathbb{Z} & & \mathbb{Z} & & 0 \\
& & & & \longrightarrow & H^1(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4]) & \longrightarrow H^1(\mathbb{Z}_4, M) \longrightarrow H^2(\mathbb{Z}_4; \mathbb{Z}) \longrightarrow \\
& & & & & 0 & \mathbb{Z}_4 \\
& & & & \longrightarrow & H^2(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4]) & \longrightarrow \dots \\
& & & & & 0 & \\
\dots & \longrightarrow & H^i(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4]) & \longrightarrow & H^i(\mathbb{Z}_4, M) & \longrightarrow & H^{i+1}(\mathbb{Z}_4; \mathbb{Z}) \longrightarrow H^{i+1}(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4]) \longrightarrow \dots \\
& & 0 & & & & 0
\end{array}$$

□

Lemma 3.4. *Let $\Upsilon \in H^*(\mathbb{Z}_4, N)$ denote an element of degree 1 such that $2\Upsilon = 0$. Then $H^*(\mathbb{Z}_4, N) \cong H^*(\mathbb{Z}_4; \mathbb{Z}[\mathbb{Z}_4/\mathbb{Z}_2]) \cdot \Upsilon$ as an $H^*(\mathbb{Z}_4; \mathbb{Z})$ -module.*

Proof. There is a short exact sequence of \mathbb{Z}_4 -modules

$$0 \rightarrow N \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}_4] \rightarrow L \rightarrow 0$$

where $L = \mathbb{Z}[\mathbb{Z}_4]/N$ and $\alpha(p, q) = (p, q, -p - q)$. The map α is well defined because the following diagram commutes

$$\begin{array}{ccc}
N =_{ab} \mathbb{Z} \oplus \mathbb{Z} \ni (p, q) & \xrightarrow{\alpha} & (p, q, -p - q) \in \mathbb{Z}[\mathbb{Z}_4] \\
& \downarrow \omega & \downarrow \omega \\
N =_{ab} \mathbb{Z} \oplus \mathbb{Z} \ni (q, -p) & \xrightarrow{\alpha} & (q, -p, -q, p) \in \mathbb{Z}[\mathbb{Z}_4]
\end{array}$$

The long exact sequence in group cohomology [3, Prop. 6.1, p 71] implies the result. □

The E_2 -term of the Borel construction $(X \setminus Y) \times_{\mathbb{Z}_4} E\mathbb{Z}_4$, with the $H^*(\mathbb{Z}_4; \mathbb{Z})$ -module structure, is presented in Figure 2.

6	0		$\Upsilon_{\mathbb{Z}_2}$		0		$T\Upsilon_{\mathbb{Z}_2}$		0		$T^2\Upsilon_{\mathbb{Z}_2}$		0		$T^3\Upsilon_{\mathbb{Z}_2}$																
5	0		0		0		0		0		0		0		0																
4	$I_{\mathbb{Z}_2}$		0		0		$\Lambda_{\mathbb{Z}_4}$		$T_{\mathbb{Z}_2}$		0		0		$\Lambda U_{\mathbb{Z}_4}$		$T^2_{\mathbb{Z}_2}$		0		0		$\Lambda U^2_{\mathbb{Z}_4}$		$T^3_{\mathbb{Z}_2}$		0		$\Lambda U^3_{\mathbb{Z}_4}$		
3	0		0		0		0		0		0		0		0		0		0		0		0		0		0				
2	\mathbb{Z}		0		0		0		0		0		0		0		0		0		0		0		0		0				
1	0		0		0		0		0		0		0		0		0		0		0		0		0		0				
0	$I_{\mathbb{Z}}$		0		$U_{\mathbb{Z}_4}$		0		$U^2_{\mathbb{Z}_4}$		0		$U^3_{\mathbb{Z}_4}$		0		$U^4_{\mathbb{Z}_4}$		0		$U^5_{\mathbb{Z}_4}$		0		$U^6_{\mathbb{Z}_4}$		0				
	0	1	2	3	4	5	6	7																							

Figure 2: The E_2 -term

The differentials of the spectral sequence are retrieved from the fact that the \mathbb{Z}_4 action on Ω is free. Therefore $H_{\mathbb{Z}_4}^i(\Omega; \mathbb{Z}) = 0$ for all $i > 8$. Since the spectral sequence is converging to the graded group associated with $H_{\mathbb{Z}_4}^i(\Omega; \mathbb{Z})$ this means that for $p + q > 8$ nothing survives. Thus the only non-zero second differentials are $d_2 : E_2^{2i+1,6} \rightarrow E_2^{2i+4,4}$, $d_2(T^i\Upsilon) = T^{i+1}$, $i > 0$, as displayed in Figure 3. The last remaining non-zero differentials are $d_4 : E_4^{2i+1,4} \rightarrow E_4^{2i+6,0}$, $d_6(U^i\Lambda) = U^{i+3}$, $i > 0$. Then $E_5 = E_\infty$, cf. Figure 4.

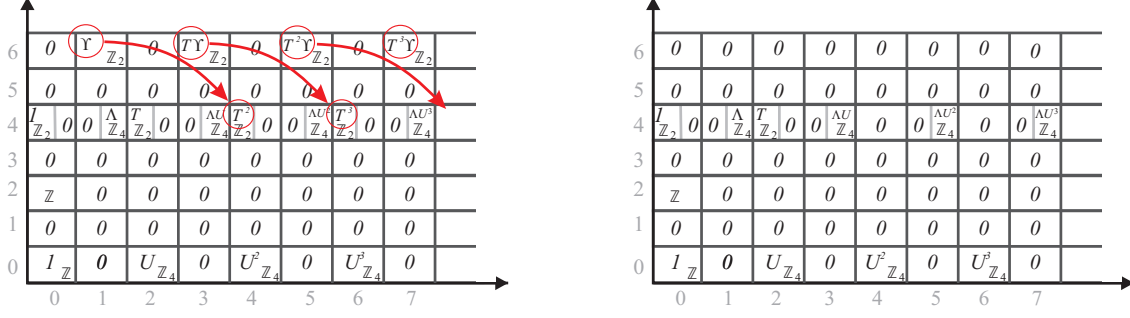


Figure 3: Differentials in E_2 and E_3 -terms

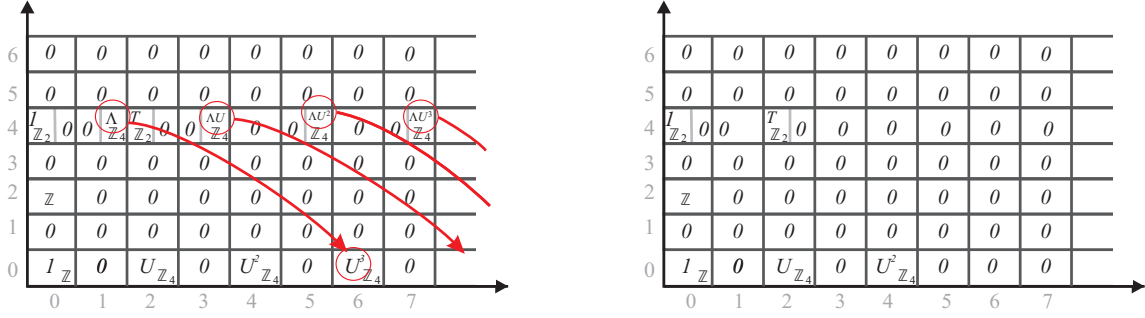


Figure 4: Differentials in E_4 and E_5 -terms

3.4 The index of Ω

The conclusion $d_6(\Lambda) = U^3$ implies that

$$\text{Index}_{\mathbb{Z}_4, \mathbb{Z}} \Omega = \langle U^3 \rangle.$$

Since the generator $2U^2$ of the $\text{Index}_{\mathbb{Z}_4, \mathbb{Z}} S(U_4 \times U_2)$ is not contained in the $\text{Index}_{\mathbb{Z}_4, \mathbb{Z}} \Omega$ it follows that **there is no equivariant map** $\Omega \rightarrow S(U_4 \times U_2)$. This concludes the proof of Theorem 2.2.

4 Concluding remarks

4.1 The \mathbb{F}_2 -index

Let $H^*(\mathbb{Z}_4, \mathbb{F}_2) = \mathbb{F}_2[e, u]/e^2$, $\deg(e) = 1$, $\deg(u) = 2$. The homomorphism of coefficients $j : \mathbb{Z} \rightarrow \mathbb{F}_2$, $j(1) = 1$, induces a homomorphism in group cohomology $j^* : H^*(\mathbb{Z}_4; \mathbb{Z}) \rightarrow H^*(\mathbb{Z}_4, \mathbb{F}_2)$ given by $j^8(U) = u$ (compare [2, Section 4.5.2]).

The \mathbb{F}_2 -index of the configuration space Ω is

$$\text{Index}_{\mathbb{Z}_4, \mathbb{F}_2} \Omega = \langle eu^2, u^3 \rangle.$$

This can be obtained in a similar fashion as we obtained the index with \mathbb{Z} -coefficients in Section 3.3. The relevant E_2 -term of the Serre spectral sequence of the fibration $\Omega \rightarrow \Omega \times_{\mathbb{Z}_4} \mathbb{E}\mathbb{Z}_4 \rightarrow \mathbb{B}\mathbb{Z}_4$ is described in Figure 5.

The \mathbb{F}_2 -index of the sphere $S(U_4 \times U_2)$ is generated by the j^* image of the generator $2U^2$ of the index with \mathbb{Z} -coefficients $\text{Index}_{\mathbb{Z}_4, \mathbb{Z}} S(U_4 \times U_2)$. Since $j^*(2U^2) = 0$ the index $\text{Index}_{\mathbb{Z}_4, \mathbb{F}_2} S(U_4 \times U_2)$ is trivial. Therefore, for our problem **no conclusion can be obtained from the study of the \mathbb{F}_2 -index**. The same observation holds even when the complete group D_8 is used. The \mathbb{F}_2 -index of the sphere $S(U_4 \times U_2)$ would be generated by $xyw = 0 \in H^*(D_8; \mathbb{F}_2)$, in the notation of [2].

6	1	t	t^2	t^3	t^4	t^5	t^6	t^7									
5	0	0	0	0	0	0	0	0									
4	1	e	t	u	t^2	eu	t^3	u^2	t^4	eu^2	t^5	u^3	t^6	eu^3	t^7	u^4	
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
2	\mathbb{F}_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	1	e	u	eu	u^2	eu^2	u^3	eu^3									
	0	1	2	3	4	5	6	7									

Figure 5: E_2 -term with \mathbb{F}_2 -coefficients

4.2 The square peg problem

The method of configuration spaces can also be set up for the continuous square peg problem. Following the ideas presented in Section 2, taking for X the product $S^1 \times S^1 \times S^1 \times S^1$, for Y the subspace $Y = \{(x, y, x, y) \mid x, y \in S^1\}$ and for the configuration space $\Omega = X \setminus Y$, the square peg problem can be related to the question of the existence of a D_8 -equivariant map $\Omega \rightarrow S(U_4 \times U_2)$. The Fadell–Hussein indexes can be retrieved:

$$\text{Index}_{\mathbb{Z}_4, \mathbb{Z}} \Omega = \langle U^2 \rangle \quad \text{and} \quad \text{Index}_{\mathbb{Z}_4, \mathbb{Z}} S(U_4 \times U_2) = \langle 2U^2 \rangle,$$

but since $\text{Index}_{\mathbb{Z}_4, \mathbb{Z}} \Omega \supseteq \text{Index}_{\mathbb{Z}_4, \mathbb{Z}} S(U_4 \times U_2)$ the result does not yield any conclusion. The same can be done for the complete symmetry group D_8 , explicitly $\text{Index}_{D_8, \mathbb{Z}} S(U_4 \times U_2) = \langle 2\mathcal{W} \rangle$ and $\mathcal{W} \in \text{Index}_{D_8, \mathbb{Z}} \Omega$.

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